Computing a common preference vector in a complex multi-actor and multi-group decision system in Analytic Hierarchy Process context

Pietro Amenta, Department of Law, Economics, Management and Quantitative Methods, University of Sannio, Piazza Arechi II, Benevento, Italy;

Alessio Ishizaka, Faculty of Business & Law, University of Portsmouth, United Kingdom;

Antonio Lucadamo, Department of Law, Economics, Management and Quantitative Methods, University of Sannio, Piazza Arechi II, Benevento, Italy;

Gabriella Marcarelli, Department of Law, Economics, Management and Quantitative Methods, University of Sannio, Piazza Arechi II, Benevento, Italy;

Vijay Vyas, Faculty of Business & Law, University of Portsmouth, United Kingdom

Abstract

The Analytic Hierarchy Process (hereafter AHP) is a popular multi-criteria decision-making technique. The extant AHP literature usually depicts the geometric mean or the arithmetic mean as a measure of aggregation to process group decisions. However, both these measures are subject to the influence of extreme opinions, and aggregations based on them may not accurately portray the true group preference. In this paper, we propose the Common Priority Vector Procedure, which accentuates the majority group preference and diminishes the influence of extreme individual opinions. The method has been further extended to deal with multi-actor, multi-criteria and multi-group decisions. The development of Common Priority Vector Procedure, presented here, has been motivated by a real case study presented towards the end of the paper.

Keywords: Decision analysis, AHP, Group decision, Common Priority Vector Procedure, multi-criteria decision making.

1. Introduction

Good decisions are key to robust business performance and several multi-criteria decision analysis (MCDA) methods to facilitate them have been developed recently [21, 26]. However, these decisions are usually not taken by a single decision-maker (*DM*) but by a group of them, who may have different personal preferences and objectives. To address this, MCDA methods have been adapted to facilitate group decisions. While doing so, two questions need to be dealt with: how to aggregate individual preferences and how to weight the preference of each individual.

Within AHP [37] several techniques to process group decisions have been developed [22]. They can be separated into either a deterministic or a probabilistic context [32]. The deterministic is the most popular case. Within this context, techniques can be grouped in three main streams: consistency based aggregation, aggregation at the individual judgment level (AIJ) and aggregation at the individual priority level (AIP) [24]. The consistency consensus matrix contains a range or interval of values that permits to have all decision-makers simultaneously consistent. The main limitation is that

on many occasion such a matrix is not possible [2] Within AIJ, the geometric mean method (GMM) is adopted because it preserves the reciprocal property [1]. However, Ramanathan and Ganesh [36] provide an example where the Pareto optimality (i.e. if all group members prefer A to B, then the group decision should prefer A) is not satisfied with the GMM. Van Den Honert and Lootsma [41] argue that this violation could be expected because the pairwise assessments constitute compromise of group members' individual assessments and therefore often generate a choice that does not truly represent collective opinion of the group members. Madu and Kuei [30] and then Saaty and Vargas [39] introduce a measure of the dispersion of judgments to address this issue. If the group is not homogeneous, further discussions are also required to reach a consensus. However, in practice, a broad consensus among decision maker is not always possible. Recently Principal Component Analysis (hereafter PCA) has been suggested for aggregating individual judgments as it leads to a robust consensus even if opinions are dispersed [39]. However, when individual judgments are aggregated, individual preferences are lost. If this is to be prevented, early aggregation should be avoided.

AIP has the added advantage of being independent of the methods used to derive the priorities (e.g. indirect elicitation with pairwise comparisons, direct evaluation of scores, etc). However, the arithmetic mean (as well as the geometric mean) is usually strongly subject to the influence of values that deviate from the majority. We highlight that this effect could be strongly amplified using a non-appropriate weights distribution for the *DMs* in the aggregation of the priority vectors. Main objection against using these automatic aggregation tools is that they could lead to a consensus priority vector not reflecting the majority of the priorities and, in extreme cases, would be completely discordant. To avoid this, we propose an aggregation at the priorities level by using a new procedure of aggregation named *Common Priority Vector Procedure* (hereafter CPVP) for aggregating the priorities in a common (consensus) preference vector.

The starting point is the priority vectors of *DMs*, obtained by computing the principal eigenvector of the positive reciprocal pairwise comparison judgment matrix [38]. The aim of this paper is to introduce a procedure of computing a common priority vector that mitigates the sensitivity of the arithmetic mean and represents the majority of the priorities. The common priority vector takes into account the preferences of a majority of *DMs*. We point out that this does not usually occur when we aggregate at the individual priority level (AIP) [24]. In addition, we highlight that the impact of this feature on the common priority vector can however be mitigated by using suitable weights for the *DMs*. These weights are introduced by a super-*DM* and can be usually computed by a pairwise DM comparison matrix.

Our new procedure is developed through a solution of an optimal criterion and gives interesting information about the misalignment of a single decision maker corresponding to that common vector. We demonstrate the practical value of this approach through its application to a real group decision in a family owned business, which wanted to adopt a participative decision process without deviating too much from its extant man-

agement style.

The rest of the paper is organised as follow. Basic notations are introduced in Section 2. In Section 3, the CPVP is proposed for deriving a common preference vector in the case of a multi-criteria or multi-actor problem and, then, it is extended to the cases of two or more groups as well as to the case of Multiple Components (hereafter MC-CPVP). In Sections 4 and 5, some numerical examples and a case study are presented. Finally, in Section 6, concluding remarks are provided.

2. Basic notations

Let us consider the following notation:

- $\mathcal{A} = \{A_1, A_2, \dots, A_n\}$ is a set of n alternatives;
- $\mathcal{L} = \{CP_1, CP_2, \dots, CP_G\}$ is a set of G sub-goals (hereafter components) and $\pi_{(g)}$ represents the relative weight of the g -th component CP_g ($g = 1, \dots, G$) with $\sum_{g=1}^G \pi_{(g)} = 1$;
- ${}^g\mathcal{P}_L = \{P_1, P_2, \dots, P_{L^{(g)}}\}$ is a set of $L^{(g)}$ groups of DM s for the g -th component CP_g , and ϕ_{l_g} ($l_g = 1, \dots, L^{(g)}$), with $\sum_{l_g=1}^{L^{(g)}} \phi_{l_g} = 1$, is the weight of the l_g -th group P_{l_g} in the g -th component;
- ${}^{l_g}\mathcal{Q}_K = \{DM_1^{(l_g)}, DM_2^{(l_g)}, \dots, DM_{K^{(l_g)}}^{(l_g)}\}$ is a set of $K^{(l_g)}$ DM s of the l_g -th group P_{l_g} , and $\alpha_k^{(l_g)}$ ($k^{(l_g)} = 1, \dots, K^{(l_g)}$) with $\sum_{k^{(l_g)}} \alpha_k^{(l_g)} = 1$, reflects the positive weight of the $k^{(l_g)}$ -th DM in the l_g -th group P_{l_g} in the g -th component.
- ${}^g\mathcal{C}_M = \{CM_1, CM_2, \dots, CM_{M^g}\}$ is a set of M^g criteria and $\beta_{m_g}^{[k^{(l_g)}]}$ ($m_g = 1, \dots, M^g$) with $\sum_{m_g=1}^{M^g} \beta_{m_g}^{[k^{(l_g)}]} = 1$, is the weight reflecting the relative importance of the m_g -th criterion for the $k^{(l_g)}$ -th DM in the l_g -th group P_{l_g} of the g -th component CP_g .

The $L^{(g)}$ groups of DM s for the g -th component CP_g have to evaluate the n alternatives with respect to the M^g criteria. The global priority vector will be formed by taking into account the weights $\pi_{(g)}$, ϕ_{l_g} , $\alpha_k^{(l_g)}$ and $\beta_{m_g}^{[k^{(l_g)}]}$. Figure 1 highlights the AHP scheme for the g -th component and relative notation.

3. The CPVP methodology

3.1. Preliminaries

Let us consider a complex AHP decision problem structured at nested decisional levels: multi-components, multi-groups, multi- DM s and multi-criteria, where each has a weight allocated. Without loss of generality, Figure 1 shows an example of an AHP decision structure with a single component ($G = 1$), $L^{(1)} = 2$ groups with $K^{(l_1)} = 3$

Figure 1. AHP scheme for the g -th component CP_g and relative notation.

and $K^{(l_2)} = 2$ DMs respectively, $M^{(1)} = 3$ criteria ${}^1C_M = \{CM_1, CM_2, CM_3\}$, where each one has a weight allocated: $\phi_{(l_1)}$ (with $l_1 = 1, 2$); $\alpha_k^{(l_1)}$ (with $k^{(l_1)} = 1, \dots, 3$), $\alpha_k^{(l_2)}$ (with $k^{(l_2)} = 1, 2$); $\beta_{m_1}^{[k^{(l_1)}]}$ (with $m_1 = 1, \dots, 3$ and $k^{(l_1)} = 1, \dots, 3$), $\beta_{m_1}^{[k^{(l_2)}]}$ (with $m_1 = 1, \dots, 3$ and $k^{(l_2)} = 1, 2$). Finally, the last level contains $n = 4$ alternatives. Dotted rectangles in Figure 2 highlight preferences aggregation problems. These problems have been widely studied in literature (e.g. [18, 22]).

Multi-criteria methods allow the decision maker to analyze different alternatives by considering several criteria and objectives. The AHP is the most popular and extensively used method that derives the preference ranking by means of pairwise comparisons and involves both qualitative and quantitative factors [42, 38]. There is significant literature on the AHP as well as on group decisions within AHP and different methods exist for aggregating judgments or priorities expressed by the members of a group [1, 16, 18, 36].

The most common methods applied for aggregating individual judgments, both in the AIJ and AIP approaches, are Weighted Arithmetic Mean Method (WAMM) and Weighted Geometric Mean Method (WGMM) [18]. This paper focuses on the AIP approach under two assumptions. The group structure is not homogeneous and the members of the group do not act as one single decision maker. Following the AIP approach, if we consider the WAMM aggregation method then the group priority of the i -th alternative is given by $q_i^{WAMM} = \sum_{k^{(l_g)}=1}^{K^{(l_g)}} q_i^{(k^{(l_g)})} \alpha_{k^{(l_g)}}^{(l_g)}$ where $q_i^{(k^{(l_g)})}$ represents the individual priority for the $k^{(l_g)}$ -th DM, in the l_g -th group P_{l_g} in the g -th component, with respect to the i -th alternative and $\alpha_k^{(l_g)}$ is the positive weight of the $k^{(l_g)}$ -th DM ($k^{(l_g)} = 1, \dots, K^{(l_g)}$). If we consider WGMM then the group priority of the i -th alter-

Figure 2. An example of an AHP decision structure with a single component, two groups of DMs (with three and two DMs, respectively), three criteria and four alternatives.

native is instead given by $q_i^{\text{WGMM}} = \prod_{k(l_g)=1}^{K(l_g)} (q_i^{(k(l_g))})^{\alpha_k^{(l_g)}}$.

The bottom-up basic MC-CPVP decision structure has essentially the same hierarchical levels of AHP but it uses the new aggregating individual priorities approach called Common Priority Vector. Another logical difference relies on the role played by the weights: in the latter method, the weights of the decisional elements (criteria and DMs) are directly computed by the pairwise comparison between all the elements of a given level, while in the former they are exogenously chosen.

The MC-CPVP decision structure can be used in cases that are more complex and where multiple groups of criteria are considered within each component as well as when several sets of groups of components are included. Obviously, these new aspects imply additional levels in the decision structure.

3.2. Common preference vector for multiple DMs by using a single criterion: the CPVP method.

Consider the following simple AHP group decision making problem with a single component ($G = 1$), a single group ($L^{(1)} = 1$) of $K^{(l_1)}$ DMs, and a single criterion ($M^1 = 1$). This choice simplifies the notation improving readability. Let $\mathcal{Q} = \{DM_1, DM_2, \dots, DM_K\}$ be then a set of DMs with α_k ($k = 1, \dots, K$) reflecting the positive weight of the k -th expert in forming the group opinion such that $\sum_{k=1}^K \alpha_k = 1$. Let $\mathbf{X}_k = ({}^k x_{ij})$ be an $n \times n$ positive reciprocal matrix, with ${}^k x_{ij} = 1/{}^k x_{ji}$ for $i, j \in \{1, \dots, n\}$ where n denotes either the number of attributes or the number of alternatives. It is to note that the reciprocity is not a necessary condition in AHP. Some studies prefers to work with non reciprocal matrices [29]. ${}^k x_{ij}$ represents the judgment expressed by the k -th DM when comparing the i th with the j th alternative according

to a single criterion. The entries of this matrix, given for the same decision problem, can be expressed by using Saaty's fundamental scale [38]. It is well known that if ${}^k x_{ij} = {}^k x_{is} {}^k x_{sj}$, for each $i, j, s \in \{1, \dots, n\}$, then judgment matrix $\mathbf{X}_k$ is perfectly consistent. If some entries are missing, techniques for completing the pairwise comparison matrices have been developed [9, 43].

The most commonly used prioritization methods are the eigenvalue method (EM) [38] and the row geometric mean method (RGMM), also called the logarithmic least squares method. Some authors [15, 25] have studied the effects of both methods and have showed that they are very similar, even if RGMM has a lower computational time. Despite this, we use EM in our approach because it provides additional information on the consistency given by the eigenvalues associated with the individual priorities and because these eigenvalues play a key role in our method.

According to EM, let $\mathbf{q}_k$ be the principal eigenvector of judgment matrix $\mathbf{X}_k$ and λ_k the associated eigenvalue such that $\mathbf{X}_k \mathbf{q}_k = \lambda_k \mathbf{q}_k$ ($k = 1, \dots, K$) with $\mathbf{q}_k^T \mathbf{q}_k = 1$. Priority vector $\tilde{\mathbf{q}}_k$ is given by $\tilde{\mathbf{q}}_k = \mathbf{q}_k / (\mathbf{1}^T \mathbf{q}_k)$ where $\mathbf{1}$ is a vector of ones such that $\mathbf{1}^T \tilde{\mathbf{q}}_k = 1$.

The main goal of our approach is to find a common priority vector describing the global preference of set $\mathcal{Q}$ in terms of the structural information given by matrices $\mathbf{X}_k$. We also suppose the existence of a common structure to the judgment matrices of rank one and, in the meantime, each matrix $\mathbf{X}_k$ provides a specific salience associated with the common priority vector. One may recognise here the basic premises of the Individual Differences Scaling [8] and Common Components and Specific Weights [34] models. However, we argue that these approaches cannot be directly applied in the AHP context due to the nature of judgment matrices.

Consider now the following positive matrix $\mathbf{W}_k = \lambda_k \mathbf{q}_k \mathbf{q}_k^T$ of unitary rank. This matrix provides the same structural information of judgment matrix $\mathbf{X}_k$ for obtaining the priority vector $\tilde{\mathbf{q}}_k$. Indeed, $\mathbf{W}_k$ provides the same real eigenvalue and eigenvector of matrix $\mathbf{X}_k$. We can use then $\mathbf{W}_k$ for $\mathbf{X}_k$ ($k = 1, \dots, K$) without losing any structural information. Figure 2 shows the conceptual scheme to perform the CPVP approach on the matrices $\mathbf{X}_k$ in order to achieve the common priority vector $\mathbf{q}$.

We suppose that all *DMs* have expressed their judgments in a rational way. This aspect is usually detected by the consistency ratio (CR) associated with the pairwise comparison matrices of *DMs*. Several consistency indices have been developed [6] and studies have proposed properties [6], axioms [10] and a general formulation [7]. It is obvious that a fully inconsistent matrix leads to poor results [27]. Therefore, inconsistent *DMs* are urged to reconsider their judgment until a satisfactory consistency threshold is reached [28]. Automatic methods for correcting the inconsistency have also been developed [35]. Since *DMs* could have judgments which are subject to change, due to their psychological states and influence of information inputs, in general, the perfect consistency condition is rarely satisfied. This is also due to the fact that our world is inconsistent by default (see, for instance, outcome of competitive sports and games). This

Figure 3. Conceptual scheme to perform the CPVP approach on the matrices $\mathbf{X}_k$.

implies that a quasi-consistent DM (that is a DM with a judgment matrix A of acceptable consistency: $CR \leq 0.1$) is usually accepted as a rational DM . In our approach we consider only DM s with $CR \leq 0.1$.

The rationale behind our approach is that the DM s may have a common priority vector even if it could be differently agreed by different experts. Some of them agree sufficiently with this common solution while others are less concordant with it. We look for a common vector $\mathbf{q}$ which reflects the values of alternatives as they are seen by the majority of DM s. A model that provides this vector can be stated as

$$\mathbf{W}_k = \tilde{\mathbf{W}}_k + \mathbf{E}_k = \tilde{\lambda}_k \mathbf{q} \mathbf{q}^T + \mathbf{E}_k$$

where $\tilde{\mathbf{W}}_k = \tilde{\lambda}_k \mathbf{q} \mathbf{q}^T$, $\tilde{\lambda}_k$ is the salience of order k associated to the common vector $\mathbf{q}$ and $\mathbf{E}_k$ is the residual matrix, with $k = 1, \dots, K$.

A vector normalized to 1 (i.e. $\sum_{i=1}^n q_i = 1$) is given by $\tilde{\mathbf{q}} = \mathbf{q} / (\mathbf{1}^T \mathbf{q})$.

The common vector $\mathbf{q}$ and the K saliencies $\tilde{\lambda}_k$ could then be found as solutions of minimization of the following loss function

$$\begin{aligned} L(\mathbf{q}, \tilde{\lambda}_k, \alpha_k) &= \sum_{k=1}^K \alpha_k \left\| \mathbf{W}_k - \tilde{\lambda}_k \mathbf{q} \mathbf{q}^T \right\|_F^2 \\ &= \sum_{k=1}^K \alpha_k \left\| \lambda_k \mathbf{q}_k \mathbf{q}_k^T - \tilde{\lambda}_k \mathbf{q} \mathbf{q}^T \right\|_F^2 \\ &= \sum_{k=1}^K \alpha_k \left\| \mathbf{W}_k - \tilde{\mathbf{W}}_k \right\|_F^2 \end{aligned} \tag{1}$$

The common vector $\mathbf{q}$ and the K saliencies $\tilde{\lambda}_k$ are then able to minimize the distance between the priority vectors $\mathbf{q}_k$ and the common vector vector $\mathbf{q}$ by using the matrices $\mathbf{W}_k$ and $\tilde{\mathbf{W}}_k$ which provide the same structural info.

The weights α_k of the K DMs in forming the group opinion are usually fixed *a priori* according to the external information (expertise, involvement, etc.) about each element of $\mathcal{Q}$. Without *a priori* information about the DMs, users usually set $\alpha_k = 1/K$, $\forall k$. It is evident that other suitable choices for α_k can be made. For example, they can be achieved by a super-DM using a pairwise DM comparison matrix.

If the α_k 's are known in the loss function then the common vector $\mathbf{q}$ and the salience $\tilde{\lambda}_k$ ($k = 1, \dots, K$) are solutions of the problem:

$$\min_{\mathbf{q}, \tilde{\lambda}_k} L(\mathbf{q}, \tilde{\lambda}_k) = \sum_{k=1}^K \alpha_k \left\| \mathbf{W}_k - \tilde{\lambda}_k \mathbf{q} \mathbf{q}^T \right\|_F^2 \quad (2)$$

where $L(\mathbf{q}, \tilde{\lambda}_k)$ can be written

$$\begin{aligned} L(\mathbf{q}, \tilde{\lambda}_k) &= \sum_{k=1}^K \alpha_k \text{tr} \left[\left(\mathbf{W}_k - \tilde{\lambda}_k \mathbf{q} \mathbf{q}^T \right) \left(\mathbf{W}_k - \tilde{\lambda}_k \mathbf{q} \mathbf{q}^T \right)^T \right] \\ &= \text{tr} \left(\sum_{k=1}^K \alpha_k \mathbf{W}_k \mathbf{W}_k^T \right) - 2 \text{tr} \left[\mathbf{q}^T \left(\sum_{k=1}^K \alpha_k \tilde{\lambda}_k \mathbf{W}_k \right) \mathbf{q} \right] + \sum_{k=1}^K \alpha_k \tilde{\lambda}_k^2 \end{aligned} \quad (3)$$

It is evident that if we consider all the salience $\tilde{\lambda}_k$ as fixed terms, then the minimum of $L(\mathbf{q}, \tilde{\lambda}_k)$ is achieved seeking the vector $\mathbf{q}$ that maximizes the quadratic form $\mathbf{q}^T \left(\sum_{k=1}^K \alpha_k \tilde{\lambda}_k \mathbf{W}_k \right) \mathbf{q}$. The solution is then provided by the first eigenvector of matrix $\sum_{k=1}^K \alpha_k \tilde{\lambda}_k \mathbf{W}_k$. Analogously, if we consider the common vector $\mathbf{q}$ *a priori* fixed, then the minimum of $L(\mathbf{q}, \tilde{\lambda}_k)$ is achieved with $\tilde{\lambda}_k = \mathbf{q}^T \mathbf{W}_k \mathbf{q}$.

In order to obtain a numerical solution, we have to implement then an iterative algorithm to solve (1). This is developed by initializing all the salience $\tilde{\lambda}_k$ to fixed values and computing every other quantity iteratively until a convergence criterion is reached or when a pre-determined number of steps is reached. The common vector $\mathbf{q}$ and all the salience $\tilde{\lambda}_k$ can then be found according to the following procedure based on the alternating least squares method:

Algorithm 1

Step 1 Starting step

1. A set of K positive weights α_k is chosen (with $\sum_{k=1}^K \alpha_k = 1$);
2. All the salience $\tilde{\lambda}_k$ are initialized to 1;
3. Set the iterative counter ξ to 1;
4. The common vector $\mathbf{q}$ is set to the first standardized eigenvector of $\sum_{k=1}^K \alpha_k \mathbf{W}_k$ and let μ_k be the associated eigenvalue;

5. Compute each salience $\tilde{\lambda}_k = \mathbf{q}^T \mathbf{W}_k \mathbf{q}$ with $k = 1, \dots, K$;
6. Compute $\eta = L(\mathbf{q}, \tilde{\lambda}_k, \alpha_k)$;
7. Set the convergence criterion variable $\varepsilon_{(\xi)}$ to η .

Step 2 *Iterative step*

8. Increase the iterative counter $\xi = \xi + 1$;
9. The common vector $\mathbf{q}$ is set to the first standardized eigenvector of $\sum_{k=1}^K \alpha_k \tilde{\lambda}_k \mathbf{W}_k$ and let μ be the associated eigenvalue;
10. The salience $\tilde{\lambda}_k$ is given by $\tilde{\lambda}_k = \mathbf{q}^T \mathbf{W}_k \mathbf{q}$ with $k = 1, \dots, K$;
11. Compute $\eta = L(\mathbf{q}, \tilde{\lambda}_k, \alpha_k)$;
12. Set the value of the convergence criterion variable $\varepsilon_{(\xi)}$ to η .

Step 3 *Convergence step*

13. If $\{|\varepsilon_{(\xi)} - \varepsilon_{(\xi-1)}|/\varepsilon_{(\xi)} \leq 10^{-6} \text{ or if } \xi = 50\}$, then $\{\text{go to step 4}\}$, else $\{\varepsilon_{(\xi)} = \varepsilon_{(\xi-1)} \text{ and go to step 2}\}$.

Step 4 *Exit*

14. Compute the common priority vector $\tilde{\mathbf{q}} = \mathbf{q}/(\mathbf{1}^T \mathbf{q})$.

Step 5 *End.*

Finally, it is worth noting that it is possible to obtain the common priority vector $\tilde{\mathbf{q}}$ as solution of $\max_{\mathbf{q}} \sum_{k=1}^K \alpha_k (\mathbf{q}^T \mathbf{W}_k \mathbf{q})^2$ subject to the constraint $\|\mathbf{q}\|^2 = 1$ with $\tilde{\mathbf{q}} = \mathbf{q}/(\mathbf{1}^T \mathbf{q})$ [23]. This implies that $\mathbf{q}$ is also equal to the first eigenvector of matrix $\mathbf{P}\mathbf{P}^T$ where $\mathbf{P} = \mathbf{Q}\mathbf{A}\mathbf{L}$ with $\mathbf{Q} = [\mathbf{q}_1, \dots, \mathbf{q}_K]$, $\mathbf{A} = \text{diag}(\sqrt{\alpha_1}, \dots, \sqrt{\alpha_K})$ and $\mathbf{L} = \text{diag}(\sqrt{\tilde{\lambda}_1}, \dots, \sqrt{\tilde{\lambda}_K})$.

We highlight that if all DMs provide the same preference vector then we have $\tilde{\lambda}_k = \mu$ with $k = 1, \dots, K$. Therefore the quantity $|\tilde{\lambda}_k - \mu|$ measures the distance of the DM to the majority. Similar result can be obtained by computing the correlation between the common vector and the DM's preference vectors.

3.3. The use of the Frobenius' norm in AHP

Let $\mathbf{A}$ be the difference between $\mathbf{W}_k$ and $\tilde{\lambda}_k \mathbf{q} \mathbf{q}^T$ in loss function (1), where $\|\mathbf{A}\|_F = (\sum_{i=1}^n \sum_{j=1}^n a_{ij}^2)^{1/2} = (\text{tr}(\mathbf{A}\mathbf{A}^T))^{1/2}$ denotes the Frobenius norm (or Hilbert-Schmidt norm) that is the Euclidean norm on $\mathbb{R}^{n \times n}$ and comes from the Frobenius inner product on the space of all matrices. $\text{tr}(\mathbf{A}) = \sum_{i=1}^n a_{ii}$ is the trace operator of an n -by- n square matrix $\mathbf{A}$.

The Frobenius' norm measures the distance between two given matrices. It has also the useful property of being invariant under rotations and the square of the norm is the sum of the eigenvalues of the matrix. These and other properties [20] lead Frobenius'

norm to be the most used standard matrix norm in numerical linear algebra, because of its simplicity with respect to other matrix norms leading to different results. For example, the popular L_1 norm $\|\mathbf{A}\|_1 = \max_j \sum_i |a_{ij}|$, that is very useful if a set of data contains outliers, is not differentiable and so it is not attractive to use in an optimization setting because it is computationally more expensive. The Frobenius' norm instead has a closed form solution and is attractive. Since we look for the solutions to reach the minimum of the loss function (1) we measure the goodness of the approximation for each k (that is $\mathbf{A}$ the difference between $\mathbf{W}_k$ and $\tilde{\lambda}_k \mathbf{q} \mathbf{q}^T$) using the norm that arises from the inner product of matrices (the Frobenius' norm). It is evident that if we measure the goodness of the approximation in another way then the final result of loss function (1) will be different.

The choice of the Frobenius' norm was also supported by its reputation and use in AHP literature. Gass and Rapsak [19] used this norm as basic tool of their SVD method and the theory of the low rank approximation of a pairwise comparison matrix as an alternative approach to EM. Lin et al. [28] introduced the Frobenius norm in their approach for deriving the priority vector from an inconsistent pairwise comparison matrix through the nearest consistent matrix and experts' judgments. The Frobenius norm was also at the heart of a linearization technique, proposed by Benitez et al. [3], that provides in AHP the closest consistent matrix to a given inconsistent matrix using orthogonal projection in a linear space. Benítez et al. [4] use the Frobenius' norm to achieve consistency in AHP when adding a new element to the decision process or when withdrawing an obsolete criterion under the dynamic input mode assumption. Farkas [17] discusses the development and use of a recursive rank-one residue iteration to balancing pairwise comparison matrices. He achieves the result by using the Newton-Kantorovich method [33] and minimizing the Frobenius's norm. Bozóki and Lewis [5] used the Frobenius norm for solving the Least Squares Method problem [12] in AHP for 3×3 and 4×4 matrices. Finally, it does not seem to us that there is use of any other matrix norms in AHP literature, perhaps because of their lack of attractiveness due to the computational difficulty and application in AHP context.

3.4. Some numerical examples

In this section, we provide three examples to illustrate the CPVP method and to compare it with classical WAMM and WGMM procedures.

Example 1

Let us consider a group of decision makers, each of which is of tolerable inconsistency ($CR < 0.1$) and faces a choice among four alternatives (Fig. 4).

As we can see in next table 1, it is easy to see that DM_1 , DM_2 , DM_3 and DM_4 prefer alternative 1, whereas DM_5 prefers the fourth alternative. The global preference

$$\begin{aligned}
X^{DM_1} &= \begin{pmatrix} 1 & 4 & 6 & 7 \\ 1/4 & 1 & 3 & 4 \\ 1/6 & 1/3 & 1 & 2 \\ 1/7 & 1/4 & 1/2 & 1 \end{pmatrix} & X^{DM_2} &= \begin{pmatrix} 1 & 5 & 7 & 9 \\ 1/5 & 1 & 4 & 7 \\ 1/7 & 1/4 & 1 & 2 \\ 1/9 & 1/6 & 1/2 & 1 \end{pmatrix} & X^{DM_3} &= \begin{pmatrix} 1 & 3 & 5 & 8 \\ 1/3 & 1 & 4 & 5 \\ 1/5 & 1/4 & 1 & 2 \\ 1/8 & 1/5 & 1/2 & 8 \end{pmatrix} \\
X^{DM_4} &= \begin{pmatrix} 1 & 4 & 5 & 6 \\ 1/4 & 1 & 3 & 3 \\ 1/5 & 1/3 & 1 & 2 \\ 1/6 & 1/3 & 1/2 & 1 \end{pmatrix} & X^{DM_5} &= \begin{pmatrix} 1 & 1/4 & 1/7 & 1/9 \\ 4 & 1 & 1/3 & 1/4 \\ 7 & 3 & 1 & 1/3 \\ 9 & 4 & 3 & 1 \end{pmatrix}
\end{aligned}$$

Figure 4. Pairwise comparison matrices for a group of five *DMs*

vectors (see table 1), computed by WAMM, WGMM and CPVP, show that the alternative 1 is preferred to the others. All *DMs* are assumed equally important in the aggregation of priorities. It is worth highlighting that the value assigned to the first alternative by CPVP is higher than those calculated with the other methods, because this procedure provides global values close to those expressed by the majority of *DMs* and is not influenced by extreme ones. This is also evident from the values assigned to the second, the third and the fourth alternative. In fact, according to CPVP, alternative 4 comprises the least preferred alternative and it seems a logical consequence if we consider the order of the preferences of the individual actors. The WAMM procedure instead, give higher importance to the high value assigned to alternative 4 by *DM*₅ and so the order in the global preference vector is different.

Alternatives				
	1	2	3	4
<i>DM</i> ₁	0.61682294	0.2238286	0.09723423	0.06211421
<i>DM</i> ₂	0.65255916	0.2246692	0.07622615	0.04654546
<i>DM</i> ₃	0.57046330	0.2771049	0.09590976	0.05652207
<i>DM</i> ₄	0.59700564	0.2217266	0.10844249	0.07282524
<i>DM</i> ₅	0.04253483	0.1281271	0.27707343	0.55226460
WAMM	0.49078770	0.2145511	0.13262689	0.16203426
WGMM	0.46149389	0.2694835	0.15029197	0.11873064
CPVP	0.60553104	0.2362991	0.09568448	0.06248537

Table 1

Individual and global priorities

As stated in the previous sections, the common priority vector procedure mitigates the sensitivity of the arithmetic mean and represents the majority of the priorities. This feature could be mitigated by applying suitable weights, assigned by a super-*DM* or derived by pairwise comparisons.

Let us consider for the previous *DMs* three different weights distributions: they are chosen to highlight the behaviour of the method, but in real analyses can be defined by the classical AHP procedure (as it has been done in the application in section 5).

- Distribution 1: $w_1 = (0.15, 0.15, 0.15, 0.15, 0.4)$. In this case the sum of weights of the first four DMs is higher than the value assigned to the fifth DM . Table 2 shows that WAMM is sensitive to the priorities associated with the DM_5 expressing opinions different from the majority; indeed the global priority vector for WAMM shows that the first alternative is the preferred one and alternatives four, two and three follow. WGMM returns a global vector that order the alternatives according to the majority of judgments expressed by the DMs , but it is evident that the intensity of the first alternative is not so high. Our procedure provides instead a common priority vector reflecting the majority of priorities. The preference order is the same of the WGMM, but the intensities are different and they are very close to the value expressed by the first four DMs .

	Alternatives			
	1	2	3	4
DM_1	0.61682294	0.2238286	0.09723423	0.06211421
DM_2	0.65255916	0.2246692	0.07622615	0.04654546
DM_3	0.57046330	0.2771049	0.09590976	0.05652207
DM_4	0.59700564	0.2217266	0.10844249	0.07282524
DM_5	0.04253483	0.1281271	0.27707343	0.55226460
WAMM	0.37549280	0.1923221	0.16977992	0.26240521
WGMM	0.30726964	0.2704073	0.21162140	0.21070165
CPVP	0.57040426	0.2295999	0.10701418	0.09298166

Table 2

Individual and global priorities with $w_1 = (0.15, 0.15, 0.15, 0.15, 0.4)$

- Distribution 2: $w_2 = (0.125, 0.125, 0.125, 0.125, 0.5)$. In this case the sum of the weights associated to the first four DMs is exactly equal to the value 0.5 assigned to DM_5 . Table 3 confirms the sensitivity of WAMM to the priorities associated with the DM_5 ; in particular, the value of the first alternative is close to the fourth alternative. The majority effect previously highlighted for our procedure is now mitigated by the high weight assigned to the DM_5 (he is as important as the other four DMs) and so the common priority vector reflects the priority associated to the most important DM .

	Alternatives			
	1	2	3	4
WAMM	0.31867260	0.1813670	0.18808985	0.31187052
WGMM	0.23801553	0.2571404	0.23838573	0.26645832
CPVP	0.04553862	0.1287185	0.27610954	0.54963331

Table 3

Individual and global priorities with $w_2 = (0.125, 0.125, 0.125, 0.125, 0.5)$

- Distribution 3: $w_3 = (0.1, 0.1, 0.1, 0.1, 0.6)$. In this structure the DM_5 has a weight higher than the sum of the other four DMs Table 4 shows that the sensitivity of WAMM is enhanced by the weights distribution; in particular, the fourth alternative

has the greatest priority. The common priority vector calculated by our procedure reflects the priority associated to the most important DM .

Alternatives				
	1	2	3	4
WAMM	0.26239338	0.1705163	0.20622546	0.36086487
WGMM	0.17823899	0.2363927	0.25960481	0.32576350
CPVP	0.04383291	0.1283828	0.27665691	0.55112742

Table 4

Individual and global priorities with $w_3 = (0.1, 0.1, 0.1, 0.1, 0.6)$

Example 2

In this example, we consider five DMs whose pairwise comparison matrices have a $CR < 0.1$ (Fig. 5). DM_2 , DM_3 and DM_5 have the same preference ranking and the values associated to each alternative are similar too: alternative 1 is preferred to alternative 2 which, in turn, is preferred to the third choice. DM_1 and DM_4 instead prefer the third alternative to the others.

The global vector resulting from the three procedures assigns the biggest value to the alternative 1 (table 5). However, once again, it is clear that the value calculated with the CPVP is not affected by the low priority assigned to the first choice by DM_1 and DM_4 , indeed, it is very close to those given by the other three actors. Some differences among the procedures are evident considering the second and the third alternatives. In fact, WAMM gives a higher value to alternative 3, whereas alternative 2 is the preferred one according to WGMM and CPVP. In this case too, the values given by CPVP to alternatives 2 and 3 are close to those given by the majority of decision makers and high correlation too is evident between the results of the CPVP and the priorities of DM_2 , DM_3 and DM_5 who have the same preference order (Table 6).

$$\begin{aligned}
 X^{DM_1} &= \begin{pmatrix} 1 & 1/7 & 1/6 \\ 7 & 1 & 1/2 \\ 6 & 2 & 1 \end{pmatrix} & X^{DM_2} &= \begin{pmatrix} 1 & 4 & 7 \\ 1/4 & 1 & 3 \\ 1/7 & 1/3 & 1 \end{pmatrix} & X^{DM_3} &= \begin{pmatrix} 1 & 3 & 4 \\ 1/3 & 1 & 2 \\ 1/4 & 1/2 & 1 \end{pmatrix} \\
 X^{DM_4} &= \begin{pmatrix} 1 & 1/2 & 1/6 \\ 2 & 1 & 1/7 \\ 6 & 7 & 1 \end{pmatrix} & X^{DM_5} &= \begin{pmatrix} 1 & 3 & 9 \\ 1/3 & 1 & 4 \\ 1/9 & 1/4 & 1 \end{pmatrix}
 \end{aligned}$$

Figure 5. Pairwise comparison matrices ($CR < 0.1$)

Example 3

In this example we increase the number of alternatives and DMs . We now have six actors who express their preferences regarding five alternatives (Fig. 6).

Alternatives			
	1	2	3
DM_1	0.0702457	0.3707310	0.55902333
DM_2	0.7049360	0.2109198	0.08414415
DM_3	0.6250131	0.2384871	0.13649980
DM_4	0.0955474	0.1440754	0.76037716
DM_5	0.6812769	0.2499474	0.06877568
WAMM	0.4356485	0.2461831	0.3181683
WGMM	0.3997492	0.3208953	0.2793556
CPVP	0.6461003	0.2358774	0.1180223

Table 5

Individual and global priorities

DMS	WAMM	WGMM	CPVP
DM1	-0.71254	-0.99897	-0.98427
DM2	0.83629	0.98846	0.99981
DM3	0.83369	0.98917	0.99989
DM4	-0.20187	-0.80498	-0.71967
DM5	0.77891	0.99852	0.99697

Table 6

Correlation among DMs and global preferences

$$\begin{aligned}
X^{DM_1} &= \begin{pmatrix} 1 & 1/4 & 1/2 & 1/3 & 1/7 \\ 4 & 1 & 8 & 1 & 2 \\ 2 & 1/8 & 1 & 1/2 & 1/4 \\ 3 & 1 & 2 & 1 & 1/3 \\ 7 & 1/2 & 4 & 3 & 1 \end{pmatrix} & X^{DM_2} &= \begin{pmatrix} 1 & 9 & 9 & 8 & 5 \\ 1/9 & 1 & 2 & 1 & 1/5 \\ 1/9 & 1/2 & 1 & 1/2 & 1/8 \\ 1/8 & 1 & 2 & 1 & 1/7 \\ 1/5 & 5 & 8 & 7 & 1 \end{pmatrix} & X^{DM_3} &= \begin{pmatrix} 1 & 9 & 9 & 5 & 1 \\ 1/9 & 1 & 1 & 1 & 1/7 \\ 1/9 & 1 & 1 & 1/7 & 1/8 \\ 1/5 & 1 & 7 & 1 & 1/6 \\ 1 & 7 & 8 & 6 & 1 \end{pmatrix} \\
X^{DM_4} &= \begin{pmatrix} 1 & 6 & 6 & 6 & 5 \\ 1/6 & 1 & 1/3 & 1 & 1/2 \\ 1/6 & 3 & 1 & 2 & 2 \\ 1/6 & 1 & 1/2 & 1 & 3 \\ 1/5 & 2 & 1/2 & 1/3 & 1 \end{pmatrix} & X^{DM_5} &= \begin{pmatrix} 1 & 1/4 & 8 & 8 & 3 \\ 4 & 1 & 8 & 8 & 4 \\ 1/8 & 1/8 & 1 & 1/2 & 1/9 \\ 1/8 & 1/8 & 2 & 1 & 1/2 \\ 1/3 & 1/4 & 9 & 2 & 1 \end{pmatrix} & X^{DM_6} &= \begin{pmatrix} 1 & 3 & 6 & 9 & 3 \\ 1/3 & 1 & 3 & 3 & 4 \\ 1/6 & 1/3 & 1 & 1/2 & 1/4 \\ 1/9 & 1/3 & 2 & 1 & 1 \\ 1/3 & 1/4 & 4 & 2 & 1 \end{pmatrix}
\end{aligned}$$

Figure 6. Pairwise comparison matrices ($CR < 0.1$)

Table 7 confirms what we have highlighted in the first two examples. In fact, the global decision, according to CPVP, is not influenced by anomalous values and the new technique also allows us to understand which agent has a behavior similar to the final choice. In table 8 the correlations confirm that the new procedure produces a global preference vector, with values close to those assigned by the majority of DMs.

The next section illustrates the CPVP method through a real case study.

3.5. Common preference vector of a single DM which evaluates n alternatives using multiple criteria: the CPVP method.

The CPVP method can also be applied to the following case with a single component ($G = 1$), a single group ($L^1 = 1$) with a single DM ($K^{l_1} = 1$), and a group of M

	Alternatives				
	1	2	3	4	5
DM_1	0.05387563	0.37200923	0.07409617	0.18014887	0.31987010
DM_2	0.59685120	0.05725899	0.03610380	0.05608439	0.25370162
DM_3	0.40391966	0.05599664	0.03774459	0.10645480	0.39588432
DM_4	0.57008783	0.06980929	0.16071607	0.11434348	0.08504334
DM_5	0.26904209	0.51303774	0.03118610	0.05131622	0.13541784
DM_6	0.48692775	0.24436339	0.05265651	0.08296006	0.13309229
WAMM	0.3868874	0.2241891	0.06537438	0.1006993	0.2228499
WGMM	0.3941013	0.1851772	0.06845795	0.1124537	0.2398098
CPVP	0.4556313	0.177045	0.06624273	0.09116508	0.2099158

Table 7

Individual and global priorities

	WAMM	WGMM	CPVP
DM1	-0.02060	-0.10816	-0.23897
DM2	0.895746	0.95166	0.96629
DM3	0.753362	0.84505	0.78304
DM4	0.72711	0.76854	0.85547
DM5	0.56251	0.41100	0.39568
DM6	0.94912	0.91661	0.95499

Table 8

Correlation among DMs and global preferences

criteria.

Let ${}^1\mathcal{C}_M = \{CM_1, CM_2, \dots, CM_M\}$ be a set of M criteria and β_m ($m = 1, \dots, M$), with $\sum_{m=1}^M \beta_m = 1$, is the weight reflecting the relative importance of the m -th criterion CM_m for this single DM . $\mathbf{X}_m = ({}^m x_{ij})$ is now an $n \times n$ positive reciprocal matrix where ${}^m x_{ij}$ represents the judgment expressed by the DM when comparing the i -th with the j -th alternative according to the m -th criterion CM_m .

We then want to find a common priority vector describing the global preference of set ${}^1\mathcal{C}_M$ in terms of the structural information given by matrices $\mathbf{X}_m$. This vector can be obtained by using the structural equivalent matrix of $\mathbf{X}_m$. Indeed, as highlighted before, matrix $\mathbf{W}_m = \lambda_m \mathbf{q}_m \mathbf{q}_m^T$ provides the same structural information of judgment matrix $\mathbf{X}_m$, where $\mathbf{q}_m$ is the principal eigenvector of judgment matrix $\mathbf{X}_m$. We then can use $\mathbf{W}_m$ for $\mathbf{X}_m$ without losing any structural information. Figure 7 shows the conceptual scheme to perform, for a generic DM_k , the CPVP approach on the matrices $\mathbf{X}_m$ in order to achieve the common priority vector $\mathbf{q}$.

The CPVP rationale in this case is that the DM may have a common priority vector while evaluating the alternatives for multiple criteria. If priorities, expressed according to the multiple criteria, are very different from one other then the common priority vector of this DM will not be very representative. This common vector could be differently agreed by taking into account other criteria. Some of these criteria may agree sufficiently with this common solution while inclusion of others may lead to a priority vector less

Figure 7. Conceptual scheme to perform the CPVP approach on the matrices X_m .

concordant with it. A common priority vector which is consistent with the most criteria will then be representative of all the priorities expressed by the *DM*. This approach thus provides a vector for evaluating alternatives that is less sensitive to the inversions of preference among alternatives, as the criterion in question varies.

According to CPVP, we look for a common vector q which reflects the values of the alternatives as they have been expressed by the majority of criteria in similar way.

A model that provides this vector can be stated as $W_m = \tilde{\lambda}_m q q^T + E_m$ where $\tilde{\lambda}_m$ is the salience of order m associated to the common vector q and E_m is the residual matrix. The common vector q and the M saliencies $\tilde{\lambda}_m$ could then be found by the loss function

$$L(q, \tilde{\lambda}_m, \beta_m) = \sum_{m=1}^M \beta_m \|W_m - \tilde{\lambda}_m q q^T\|_F^2 \quad (4)$$

The solutions can be obtained by adapting the procedure (see Algorithm 2 in Appendix) based on the alternating least squares method and introduced in subsection 3.2.

4. The AHP multi-component, multi-criteria, multi-group and multi-actor decision problem: the MC-CPVP approach.

Let us now consider a more complex AHP problem which is structured in several nested decision levels with a bottom up progression of common vectors to compute. The

bottom-up basic MC-CPVP decision structure is equivalent to the AHP one but it uses CPVP as prioritization method at each level. Figure 8 shows a MC-CPVP schematic representation with basic five levels.

The common vector of each level is given by aggregating the common vectors calculated at the underlying level. The direct application of the CPVP to each common decision problem of levels 4 and 5 provides the common vectors for those of level 3. Similarly, those of level 3 provide in similar way the common vectors for the decision problem of level 2, and so on.

Here, we point out that CPVP cannot be directly applied from level 3 to the earlier ones because their starter elements are vectors and not matrices. Indeed, the starting points of level 5 are matrices and the multi-criteria problems of level 4 are solved by CPVP according to section 3.4 providing the common vectors of level 3.

The prioritization method CPVP is instead applied starting from level 3 by computing a structural equivalent matrix for each common vector for that level. It is built using the idea used in the section 3.2. Indeed, the matrix $\mathbf{W}_{k(l_g)} = \mu_{k(l_g)} \mathbf{q}_{k(l_g)} \mathbf{q}_{k(l_g)}^T$, of unitary rank, supplies the same structural information of a hypothetical common judgment matrix which provides the same common vector $\mathbf{q}_{k(l_g)}$ and the eigenvalue $\mu_{k(l_g)}$ using EM.

For the decision problems of level 3, the matrix $\mathbf{W}_{k(l_g)}$, built from the common vector and its associated eigenvalue, can therefore be used replacing the matrices $\mathbf{W}_k$ in CPVP algorithm of section 3.2, substituting the weights α_k with $\alpha_k^{(l_g)}$. All the summation indices are consequently expressed as $k = 1, \dots, K^{(l_g)}$ in all formulas of the algorithm. The common vector $\mathbf{q}_{(l_g)}$ ($l_g = 1, \dots, L^{(g)}$) and the salience $\tilde{\mu}_{k(l_g)}$ for each element of level 2 are then solutions of the problem:

$$\min_{\mathbf{q}_{(l_g)}, \mu_{k(l_g)}} L_3(\mathbf{q}_{(l_g)}, \tilde{\mu}_{k(l_g)}) = \sum_{k(l_g)=1}^{K^{(l_g)}} \alpha_k^{(l_g)} \left\| \mathbf{W}_{k(l_g)} - \tilde{\mu}_{k(l_g)} \mathbf{q}_{(l_g)} \mathbf{q}_{(l_g)}^T \right\|_F^2 \quad (5)$$

with $\tilde{\mu}_{k(l_g)}$ being the eigenvalue associated to the common vector $\mathbf{q}_{(l_g)}$. Algorithm 3 (see Appendix) provides the procedure for achieving the common priority vector $\tilde{\mathbf{q}}_{(l_g)}$ that is solution of the generic l_g -th CPVP problem of Level 3 (Multi-DMs problem) according to (5).

The common vector $\mathbf{q}_{(g)}$ ($g = 1, \dots, G$) and the salience $\tilde{\mu}_{(l_g)}$ for each element of level 1 are instead solutions of the problem:

$$\min_{\mathbf{q}_{(g)}, \tilde{\mu}_{(l_g)}, \phi_{(l_g)}} L_2(\mathbf{q}_{(g)}, \tilde{\mu}_{(l_g)}, \phi_{(l_g)}) = \sum_{l_g=1}^{L^{(g)}} \phi_{(l_g)} \left\| \mathbf{W}_{(l_g)} - \tilde{\mu}_{(l_g)} \mathbf{q}_{(g)} \mathbf{q}_{(g)}^T \right\|_F^2 \quad (6)$$

with $\mathbf{W}_{(l_g)} = \mu_{(l_g)} \mathbf{q}_{(l_g)} \mathbf{q}_{(l_g)}^T$ and $\tilde{\mu}_{(l_g)}$ being the eigenvalue associated to the common vector $\mathbf{q}_{(g)}$. Algorithm 4 (see Appendix) provides then the procedure for achieving the

Figure 8. Schematic representation of the MC-CPVP's levels.

common priority vector $\tilde{\mathbf{q}}_{(g)}$ that is solution of the generic g -th CPVP problem of Level 2 (Multi-Groups problem) according to (6).

Finally, similar to previous levels, we can use the matrix $\mathbf{W}_{(g)} = \mu_{(g)} \mathbf{q}_{(g)} \mathbf{q}_{(g)}^T$ in the decision problem of level 1 substituting the weights α_k with π_g and the summation indices with $g = 1, \dots, G$, in all formulas of the CPVP algorithm of section 3.2. The final common priority vector $\tilde{\mathbf{q}}$ and the salience $\tilde{\mu}_g$ ($g = 1, \dots, G$) for the decision problem of level 1 are then solutions of the problem:

$$\min_{\mathbf{q}, \pi_g, \tilde{\mu}_g} L_1(\mathbf{q}, \tilde{\mu}_g, \pi_g) = \sum_{g=1}^G \pi_g \left\| \mathbf{W}_g - \tilde{\mu}_g \mathbf{q} \mathbf{q}^T \right\|_F^2 \quad (7)$$

The final global priority vector $\tilde{\mathbf{q}}$ will be given by the usual normalization formula applied to the final common vector $\mathbf{q}$: $\tilde{\mathbf{q}} = \mathbf{q} / (\mathbf{1}^T \mathbf{q})$ with $\mathbf{1}^T \tilde{\mathbf{q}} = 1$. The procedure for achieving then the overall common priority vector $\tilde{\mathbf{q}}$, solution of the CPVP problem of Level 1 (Multi-Components problem) according to (7).

5. The case study company and the knowledge transfer project

5.1. Background

The case discussed here relates to a small British company which designs and produces personalised homeowners' manuals that are sold to the property developers. The property developers buy these manuals to supply to their own clients, the new homeowners, when they purchase a property. The company writes, designs, edits and prints personalised manual content and packs it in aesthetically created high quality ring binders. In 2010, the company had a workforce of 14. During the 2007-2008 recession, the company's vulnerability, due to its dependence on a single market, was exposed. The company managed to meet this challenge through downsizing and other cost-cutting measures and remained in good health. However, in the post-recession period, it wanted to be more ambitious and aspired to enter a high growth trajectory through broadening of its portfolio of products and markets. In order to achieve these objectives, it needed to develop and implement a diversification plan. As it had no relevant experience, it entered into a knowledge transfer project (KTP) partnership with the University of Portsmouth to take advantage of its expertise. Two of the authors of this paper from the University of Portsmouth were members of the KTP team. In the first phase of the project, a resources, processes and capabilities analysis of the business was carried out in consultation with company's key stakeholders. In the second phase, an in-depth understanding of the company's current and potential products and markets was combined with the understanding of its unique resources and distinctive capabilities, to develop a diversification strategy. This project was not without its challenges as scholarly writings and research in strategic management is focussed mainly on large corporations [31]. Though recently there

has been some work on SMEs (for instance [40, 13, 14]), there is no precedent in literature that charts development and implementation of a diversification strategy for a micro enterprise such as the case study company. Another issue was that the company had few formal processes and its directors had little formal management education. The challenge was to deploy a participative approach to facilitate the company's shift from an intuitive, flexible and pragmatic style to a strategic managerial one, without diluting its entrepreneurial heritage. In order to have a broad view and an inclusive decision process, eighteen decision-makers were included from different groups (Table 9) at various stages of development of company's diversification strategy.

Group	Decision Makers
KTP members	KTP1, KTP2, KTP3, KTP4, KTP5
Company staff	CT1, CT2, CT3, CT4
Consultants	C1, C2
Business partners	B1, B2
Directors' friends & relatives	F1, F2, F3
Directors	D1, D2

Table 9
Group of decision-makers

5.2. Problem structuring

The first step was to have on-site visits to key suppliers, distributors and customers to gather vital data for the assessment of the core business process. Building on it and in consultation with the company management team, further analysis was carried out to gain an understanding of company's unique resources, which were difficult for competitors to imitate or obtain and of company's distinctive capabilities, which underpinned its sustainable competitive advantage. This analysis evolved in a strategic capability document that recorded the company's unique resources and distinctive capabilities, which were used in the next phase to identify the products and services that it can deliver and the markets that it can enter, consistent with company's core values. The potential diversification alternatives were crystallised through two brainstorming sessions. The first of these occurred at the university and was attended by a KTP Associate, one of the authors, two company directors, two other members of the KTP team and one senior academic from the university. The brainstorming panel members were given a presentation on company's existing products as well as on its core competencies and the product DNA identified in the early phase of KTP. The second brainstorming session was organised at the case study company. It was attended by the KTP Associate, one of the authors and 7 senior employees of the case study company. The objective of the first brainstorming session was to use the knowledge embedded within the university and its academic staff to provide the company some fresh and impartial insights on potential industries or markets to enter. The objective of the second brainstorming session was to

use the perceptions of the company's key employees to explore further its diversification options. Use of these two diverse sets of inputs during the brainstorming sessions enriched the diversification-option search process and provided two complementary perspectives to it. To optimise the outcome of the brainstorming process, the panel members were given the following list of potential diversification products and markets that were already identified by the company's directors.

- Souvenir cabin binders and content for luxury cruise liners.
- Hotel and local information binders for the hotel industry.
- Education/Training Folios for private schools and corporations.
- High-end business presentations documentation for tendering and other purposes.
- Customer care products, services, systems and consultancy for property developers.
- Binders for luxury boats and classic car owners.

To identify the potential diversification choices, during the brainstorming session, the panel members were asked to explore the diversification opportunities in the following:

- Companies or industries that sell high value products.
- Industries where building a close relationship with the customer is essential.
- Companies where making a great first impression is an intrinsic part of the industry culture.
- Industries where the quality of the product presentation is as important as the product itself.
- Markets or industries that require both personalised products as well as flexible lead times.
- Companies or industries that achieve high levels of customisation for their products through individually custom-made components.
- Companies that can offer better products to their customers by using the case study company's unique resources and capabilities.
- Companies, industries or individuals that intensely use customised ring binders or presentation folders.

Twenty-five diversification alternatives that emerged during these brainstorming sessions belong to three potential diversification routes:

- Related diversification route: This contains options where the case study company can use its current business model, its set of core competencies and its existing products, or the products consistent with the company product DNA.
- Unrelated diversification route: To take this route the case study company needs to develop new expertise, set of skills or business model.
- No diversification route: This comprises of options that keep the case study company in its current market.

To take a final decision on the alternatives that qualify to enter the AHP process,

each brainstorming panel member was given a scoring sheet to mark the alternatives on six questions with possible answers (Yes, Not sure, No):

- Is the product bundled with a high value product (or service)?
- Is the aesthetic quality of the product important in the proposed industry?
- Does the product require high levels of personalisation?
- Can the case study company's publishing services be used to deliver the product?
- Does the product require to be delivered just in time?
- Is establishing strong relationships with customers and suppliers important in the proposed industry?

Each Yes would score one point. Each alternative scoring in average 4 points or more would qualify to go to the second round. The alternatives that emerged at the end of this process are presented in Table 10.

Alternatives	Description
Business Folio	Personalised frame to carry and present strategic business documentation used during competitive tenders, IPOs (initial public offering), board meetings and meetings with customers/suppliers etc.
Training Folio	Personalised case to record and present personal development, staff induction, health and safety, training and conference material.
Funeral Folio	Personalised frame to record and present written information and pictures, e.g. book of remembrance.
Student and planer Folio	Personalised student diaries, record books and teacher planner.
Club and personal Folio	Personalised frame to record and present information about the history, activities, members and players of the club/society/trust.
Marine Folio	Personalised frame to carry and present customers' product or service documentation (e.g. technical specs, use and maintenance documentation, log book).
Holiday Folio	Personalised souvenir folio containing information about the holiday package (e.g. itinerary) plus an empty photo album section.

Table 10
Alternatives

The problem was then separated into a hierarchy of criteria that need to be maximised (the benefit hierarchy in Figure 9), and those that need to be minimised (the cost hierarchy in Figure 10). This separation was done because it is easier to compare criteria in the same direction than to compare criteria having conflicting objectives. It also makes the problem more manageable as two similar matrices need to be filled. Less pairwise comparison are required and it is easier to have consistent matrices. The only difference in the two hierarchies is the criteria in level 3. It should be noted that a pairwise comparison of all elements on a level is performed as regards to each element belonging to the upper level. The level 2 is however an exception as only the elements belonging to the group of decision-makers are compared separately, i.e. KTP1, KTP2, KTP3, KTP4 and KTP5 belong to one comparison matrix and C1 and C2 belong to another comparison matrix.

Figure 9. Hierarchy of benefits criteria

Figure 10. Hierarchy of costs criteria

5.3. Results

The pairwise comparisons were recorded in an individual questionnaire which was completed by all eighteen participants. Directors have assigned the weights of the groups by means of pairwise comparisons; obviously, the group of Directors has a weight greater than the other groups.

As per the procedure described in section 3, groups and global priorities for benefits and costs are obtained (see tables 11 and 12, respectively). The weights α_k and β_m used in these problems are derived from the EVs associated to pairwise matrices of DMs' groups and criteria, respectively. The global priority is calculated by taking the benefits/costs ratio and then normalizing it to 1 (Table 13).

Groups	Alternatives						
	Business Folio	Training Folio	Funeral Folio	Student Folio & Planer	Club Folio & personal organiser	Marine Folio	Holiday Folio
Directors	0.136443	0.207184	0.44711	0.486657	0.296372	0.487131	0.420218
KTP team	0.291708	0.296138	0.567571	0.181992	0.443537	0.443063	0.280927
Team	0.074824	0.318696	0.291859	0.584155	0.622833	0.26322	0.095966
Consultants	0.379296	0.304177	0.780687	0.072198	0.167072	0.303362	0.170249
Business partners	0.520632	0.329239	0.294517	0.407185	0.212591	0.550562	0.140321
Directors' friends & relatives	0.513363	0.427438	0.426998	0.325733	0.195792	0.458291	0.130238
Aggregation methods							
WAMM	0.230585	0.264944	0.493918	0.370551	0.324894	0.428552	0.295312
WGMM	0.193927	0.258668	0.473711	0.304176	0.301201	0.418134	0.260846
CPVP	0.342365	0.379840	0.458495	0.378905	0.417186	0.423704	0.176500

Table 11

Priorities for benefits hierarchy

Groups	Alternatives						
	Business Folio	Training Folio	Funeral Folio	Student Folio & Planer	Club Folio & personal organiser	Marine Folio	Holiday Folio
Directors	0.115709	0.149478	0.114549	0.329711	0.894633	0.110591	0.17274
KTP team	0.430042	0.540894	0.237527	0.250663	0.291613	0.428116	0.367322
Team	0.723309	0.335322	0.116282	0.372079	0.271616	0.172687	0.329884
Consultants	0.260031	0.303633	0.276658	0.31429	0.298498	0.567907	0.503244
Business partners	0.453493	0.339566	0.185228	0.548549	0.252408	0.462266	0.25773
Directors' friends & relatives	0.292919	0.380178	0.334936	0.259953	0.299878	0.571774	0.415996
Aggregation methods							
WAMM	0.279676	0.276586	0.173728	0.327674	0.575434	0.280172	0.288703
WGMM	0.221606	0.244142	0.159827	0.322858	0.49212	0.217293	0.263366
CPVP	0.129481	0.162955	0.125558	0.336284	0.878267	0.137511	0.193819

Table 12

Priorities for cost hierarchy

The rankings that emerge from the three approaches are different. In our approach, extremes are less pronounced (e.g. Funeral Folio is first but the gap with the second

Alternatives	MC-CPVP	WAMM	WGMM
Business Folio	0.19 (3)	0.09 (6)	0.09 (6)
Training Folio	0.16 (4)	0.10 (5)	0.11 (3)
Funeral Folio	0.26 (1)	0.32 (1)	0.31 (1)
Student Folio & Planer	0.08 (5)	0.13 (3)	0.10 (4)
Club Folio & Personal Organiser	0.03 (7)	0.06 (7)	0.07 (7)
Marine Folio	0.22 (2)	0.17 (2)	0.21 (2)
Holiday Folio	0.06 (6)	0.12 (4)	0.10 (4)

Table 13
Global priorities

alternative is smaller). In all the three rankings, the product Funeral folio, emerges as favourite diversification alternative, which makes it a robust solution. This product comprises of a personalised folio to record and present textual and pictorial information related to the deceased. It is like a book of remembrance of the departed relative. However, the aesthetic quality and richness of written as well as graphic information is expected to be much better to differentiate it from similar available products and to position it at the higher end of the value chain. In this way, all attributes that make company's current product successful and unique are applied to a different context to allow the company to enter a new market successfully.

6. Conclusion and new challenges

In this paper, we propose an aggregation procedure at priority level, which takes into account the majority group preference and the influence of extreme values in creating a consensus priority vector. CPVP is a novel approach that gives greater importance to the majority or consensual views and significantly less to the outliers. Using the priority vectors of *DMs* by means the EM approach [38], our procedure computes then a common priority vector that mitigates also the sensitivity of the arithmetic mean. The impact of the majority on the common priority vector can be however mitigated by using suitable weights for the *DMs*. These weights are introduced by a super-*DM* and can be usually computed by a pairwise *DM* comparison matrix. Our new procedure computes a priority vector that is a solution of an optimal criterion providing additional interesting information about the misalignment of a single *DM* corresponding to that common vector.

An extension of our proposal to the AHP multi-component, multi-criteria, multi-group and multi-actor decision problems is moreover performed. This CPVP extension, named MC-CPVP, has been indeed motivated by a real problem of satisfying growth aspirations of a micro enterprise utilising academic knowledge and rigor. The aim was to use scholarly insights to facilitate a shift in a small company's intuitive pragmatic approach to a strategic managerial one, without diluting its entrepreneurial heritage. The

paper is focussed on the later part of the process in which a diversification plan was shaped by an innovative modification to AHP. It combined continual stakeholder participation with methodological robustness of an amended AHP to meet successfully the desire of a small enterprise to enter a high growth path.

Decisions such as those analysed here are complex, especially if they are based on multiple criteria and involve several decision-makers from diverse groups. For the situations, in which decision-makers are unwilling or unable to revise their judgments to achieve a consensus, various techniques have been proposed in the extant literature to aggregate diverse views. However, when the dispersion in preferences is high, we risk having a middle compromise that satisfies nobody. During the course of KTP, the company used a range of contemporary management tools including modified AHP as explained in this paper. It subsequently did enter a high growth trajectory and met the primary objective of KTP. It now employs 40 people, a near 3-fold increase in employment in 6 years. As the proposed technique discussed here is generic, we believe, it can be applied, with appropriate adjustments, to other similar cases in future research. More importantly, it can be used in a wide variety of group decision making in business as well as non-business settings.

The logic approach that is at hearth of our proposal opens up to new challenges and developments, overcoming possible CPVP limitations. Indeed, starting point of our approach is the computation of the priority vectors, derived by Saaty eigenvector method. However, several prioritization methods have been introduced in literature (e.g. Saaty and Vargas 1984; Crawford and Williams 1985; Barzilai 1997; Gass and Rapsak 2004; Kou and Lin 2013). These methods can lead to different priority vectors of group. The final solution of our approach may then depend on the method used for calculating priority vectors. So, we are developing a wide simulation to verify the performance of our approach on varying the prioritization method of the CPVP starting point.

A further research should try to reduce the global majority effect of this procedure, allowing the super *DM* to choose the level of the effect of the majority on the final solution mathematically. The aim of upgrading the impact of the few DMs which are not in agreement with the other ones, could then be achieved by developing an extension of the criterion (1) which is at hearth of our approach. Indeed, we could consider the following s -powered criterion $\sum_{k=1}^K \alpha_k \left\| \mathbf{W}_k - \tilde{\lambda}_k \mathbf{q} \mathbf{q}^T \right\|_F^s$ with $s > 0$. It is evident that it subsumes our approach for $s = 2$. It could be then interesting to evaluate the impact on the majority effect varying s from 0 to ∞ . This issue is now under investigation and development.

Finally, regardless of the influence of the prioritization method on our solution, it could be also interesting to expand on our approach focusing on the preference matrices directly. The final priority vector of a group will be then provided by using a a priori chosen prioritization method on a final suitable consensus preference matrix. This issue is also under investigation.

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Appendix

Algorithm 2 Common preference vector of a single *DM* which evaluates n alternatives using multiple criteria: the CPVP method.

▷ **Settings of the procedure:**

- a single component ($G = 1$)
 - a single group ($L^1 = 1$)
 - a single *DM* ($K^{l_1} = 1$)
 - n alternatives
 - a set of M criteria with $m = 1, \dots, M$
-

Input: the number of alternatives n

Output: the common priority vector $\tilde{\mathbf{q}}$

▷ **Step 1: Starting step**

- 1: A set of M positive weights β_m is chosen (with $\sum_{m=1}^M \beta_m = 1$);
- 2: The set of the matrices $\mathbf{W}_m = \lambda_m \tilde{\mathbf{q}}_m \tilde{\mathbf{q}}_m^T$ is computed (with $m = 1, \dots, M$), where λ_m and $\tilde{\mathbf{q}}_m$ have been obtained applying the Saaty's prioritization method [37, 38] on the pairwise judgment matrix $\mathbf{X}_m$
▷ See subsection 3.4
- 3: All the salience $\tilde{\lambda}_m$ are initialized to 1;
- 4: Set the iterative counter ξ to 1;
- 5: The common vector $\mathbf{q}$ is set to the first standardized eigenvector of $\sum_{m=1}^M \beta_m \mathbf{W}_m$ and let μ_m be the associated eigenvalue;
- 6: Compute each salience $\tilde{\lambda}_m = \mathbf{q}^T \mathbf{W}_m \mathbf{q}$ with $m = 1, \dots, M$;
- 7: Compute $\eta = L(\mathbf{q}, \tilde{\lambda}_m, \beta_m)$; ▷ See loss function (4)
- 8: Set the convergence criterion variable $\varepsilon_{(\xi)}$ to η .

▷ **Step 2: Iterative step**

- 9: Increase the iterative counter $\xi = \xi + 1$;
- 10: The common vector $\mathbf{q}$ is set to the first standardized eigenvector of $\sum_{m=1}^M \beta_m \tilde{\lambda}_m \mathbf{W}_m$ and let μ be the associated eigenvalue;
- 11: The salience $\tilde{\lambda}_m$ is given by $\tilde{\lambda}_m = \mathbf{q}^T \mathbf{W}_m \mathbf{q}$ with $m = 1, \dots, M$;
- 12: Compute $\eta = L(\mathbf{q}, \tilde{\lambda}_m, \beta_m)$; ▷ See loss function (4)
- 13: Set the value of the convergence criterion variable $\varepsilon_{(\xi)}$ to η .

▷ **Step 3: Convergence step**

- 14: **if** $\{|\varepsilon_{(\xi)} - \varepsilon_{(\xi-1)}|/\varepsilon_{(\xi)} \leq 10^{-6} \text{ or } \xi = 50\}$ **then** {Go to Step 4}
- 15: **else** {Set $\varepsilon_{(\xi)} = \varepsilon_{(\xi-1)}$ and Go to Step 2}

▷ **Step 4: Exit**

- 16: Compute the common priority vector $\tilde{\mathbf{q}} = \mathbf{q}/(\mathbf{1}^T \mathbf{q})$.

▷ **Step 5: End**

Algorithm 3 The AHP multi-components, multi-criteria, multi-groups and multi-actors decision problem: the MC-CPVP approach. Multi-DMs problem.

▷ **Procedure for the generic l_g -th CPVP problem of LEVEL 3:**

- G components ($g = 1, \dots, G$)
 - $L^{(g)}$ groups for each component ($l_g = 1, \dots, L^{(g)}$)
 - $K^{(l_g)}$ DM $_k^{(l_g)}$ in each group ($k = 1, \dots, K^{(l_g)}$)
 - $K^{(l_g)}$ DM weights $\alpha_k^{(l_g)}$ in each group with $\sum_{k=1}^{K^{(l_g)}} \alpha_k^{(l_g)} = 1$.
-

▷ Setting examples for the notation:

- If $g = 1$ then $l_1 = 1, \dots, L^{(1)}$; DM $_k^{(l_1)}$ and $\alpha_k^{(l_1)}$ with $k = 1, \dots, K^{(l_1)}$
 - If $g = G$ then $l_G = 1, \dots, L^{(G)}$; DM $_k^{(l_G)}$ and $\alpha_k^{(l_G)}$ with $k = 1, \dots, K^{(l_G)}$
-

Input: The set of $K^{(l_g)}$ CPVP priority vectors $\tilde{\mathbf{q}}_{k^{(l_g)}}$ of the l_g -th group members

Output: the common priority vector $\tilde{\mathbf{q}}$ of l_g -th group

▷ **Step 1: Starting step**

- 1: A set of $K^{(l_g)}$ positive weights $\alpha_k^{(l_g)}$ is chosen (with $\sum_{k=1}^{K^{(l_g)}} \alpha_k^{(l_g)} = 1$);
- 2: The set of the matrices $\mathbf{W}_{k^{(l_g)}} = \lambda_{k^{(l_g)}} \tilde{\mathbf{q}}_{k^{(l_g)}} \tilde{\mathbf{q}}_{k^{(l_g)}}^T$ is formed (with $k = 1, \dots, K^{(l_g)}$), where $\lambda_{k^{(l_g)}}$ and $\tilde{\mathbf{q}}_{k^{(l_g)}}$ have been computed in Levels 4 and 5;
- 3: All the salience $\tilde{\mu}_{k^{(l_g)}}$ are initialized to 1;
- 4: Set the iterative counter ξ to 1;
- 5: The common vector $\mathbf{q}_{(l_g)}$ is set to the first standardized eigenvector of $\sum_{k=1}^{K^{(l_g)}} \alpha_k^{(l_g)} \mathbf{W}_{k^{(l_g)}}$ and let $\mu_{k^{(l_g)}}$ be the associated eigenvalue;
- 6: Compute each salience $\tilde{\mu}_{k^{(l_g)}} = \mathbf{q}_{(l_g)}^T \mathbf{W}_{k^{(l_g)}} \mathbf{q}_{(l_g)}$ with $k = 1, \dots, K^{(l_g)}$;
- 7: Compute $\eta = L_3(\mathbf{q}_{(l_g)}, \tilde{\mu}_{k^{(l_g)}}, \alpha_k^{(l_g)})$; ▷ See loss function (5)
- 8: Set the convergence criterion variable $\varepsilon_{(\xi)}$ to η .

▷ **Step 2: Iterative step**

- 9: Increase the iterative counter $\xi = \xi + 1$;
- 10: The common vector $\mathbf{q}_{(l_g)}$ is set to the first standardized eigenvector of $\sum_{k=1}^{K^{(l_g)}} \alpha_k^{(l_g)} \mathbf{W}_{k^{(l_g)}}$ and let $\mu_{k^{(l_g)}}$ be the associated eigenvalue;
- 11: Compute each salience $\tilde{\mu}_{k^{(l_g)}} = \mathbf{q}_{(l_g)}^T \mathbf{W}_{k^{(l_g)}} \mathbf{q}_{(l_g)}$ with $k = 1, \dots, K^{(l_g)}$;
- 12: Compute $\eta = L_3(\mathbf{q}_{(l_g)}, \tilde{\mu}_{k^{(l_g)}}, \alpha_k^{(l_g)})$; ▷ See loss function (5)
- 13: Set the value of the convergence criterion variable $\varepsilon_{(\xi)}$ to η .

▷ **Step 3: Convergence step**

- 14: **if** $\{|\varepsilon_{(\xi)} - \varepsilon_{(\xi-1)}|/\varepsilon_{(\xi)} \leq 10^{-6} \text{ or } \xi = 50\}$ **then** {Go to Step 4}
- 15: **else** {Set $\varepsilon_{(\xi)} = \varepsilon_{(\xi-1)}$ and Go to Step 2}

▷ **Step 4: Exit**

- 16: Compute the common priority vector $\tilde{\mathbf{q}}_{(l_g)} = \mathbf{q}_{(l_g)} / (\mathbf{1}^T \mathbf{q}_{(l_g)})$.

▷ **Step 5: End**

Algorithm 4 The AHP multi-components, multi-criteria, multi-groups and multi-actors decision problem: the MC-CPVP approach. The Multi-Groups problem.

▷ **Procedure for the generic g -th CPVP problem of LEVEL 2:**

- G components ($g = 1, \dots, G$)
 - $L^{(g)}$ groups for each component ($l_g = 1, \dots, L^{(g)}$)
 - $L^{(g)}$ weights $\phi_{(l_g)}$ for each group with $\sum_{l_g=1}^{L^{(g)}} \phi_{(l_g)} = 1$.
-

▷ Setting examples for the notation:

- If $g = 1$ then $\phi_{(l_1)}$ with $l_1 = 1, \dots, L^{(1)}$
 - If $g = G$ then $\phi_{(l_G)}$ with $l_G = 1, \dots, L^{(G)}$ with $\sum_{l_G=1}^{L^{(G)}} \phi_{(l_G)} = 1$.
-

Input: The set of $L^{(g)}$ CPVP priority vectors $\tilde{\mathbf{q}}_{(l_g)}$ of the l_g -th group of g -th component

Output: the common priority vector $\tilde{\mathbf{q}}_{(g)}$ of g -th component

▷ **Step 1: Starting step**

- 1: A set of $L^{(g)}$ positive weights $\phi_{(l_g)}$ is chosen (with $\sum_{l_g=1}^{L^{(g)}} \phi_{(l_g)} = 1$);
- 2: The set of the matrices $\mathbf{W}_{(l_g)} = \tilde{\mu}_{(l_g)} \tilde{\mathbf{q}}_{(l_g)} \tilde{\mathbf{q}}_{(l_g)}^T$ is formed (with $l_g = 1, \dots, L^{(g)}$), where $\mu_{(l_g)}$ and $\tilde{\mathbf{q}}_{(l_g)}$ have been computed in Level 3;
- 3: All the salience $\tilde{\mu}_{(l_g)}$ are initialized to 1;
- 4: Set the iterative counter ξ to 1;
- 5: The common vector $\mathbf{q}_{(g)}$ is set to the first standardized eigenvector of $\sum_{l_g=1}^{L^{(g)}} \phi_{(l_g)} \mathbf{W}_{(l_g)}$ and let $\mu_{(l_g)}$ be the associated eigenvalue;
- 6: Compute each salience $\tilde{\mu}_{(l_g)} = \mathbf{q}_{(g)}^T \mathbf{W}_{(l_g)} \mathbf{q}_{(g)}$ with $l_g = 1, \dots, L^{(g)}$;
- 7: Compute $\eta = L_2(\mathbf{q}_{(g)}, \mu_{(l_g)}, \phi_{(l_g)})$; ▷ See loss function (6)
- 8: Set the convergence criterion variable $\varepsilon_{(\xi)}$ to η .

▷ **Step 2: Iterative step**

- 9: Increase the iterative counter $\xi = \xi + 1$;
- 10: The common vector $\mathbf{q}_{(g)}$ is set to the first standardized eigenvector of $\sum_{l_g=1}^{L^{(g)}} \phi_{(l_g)} \mathbf{W}_{(l_g)}$ and let $\mu_{(l_g)}$ be the associated eigenvalue;
- 11: Compute each salience $\tilde{\mu}_{(l_g)} = \mathbf{q}_{(g)}^T \mathbf{W}_{(l_g)} \mathbf{q}_{(g)}$ with $l_g = 1, \dots, L^{(g)}$;
- 12: Compute $\eta = L_2(\mathbf{q}_{(g)}, \mu_{(l_g)}, \phi_{(l_g)})$; ▷ See loss function (6)
- 13: Set the value of the convergence criterion variable $\varepsilon_{(\xi)}$ to η .

▷ **Step 3: Convergence step**

- 14: **if** $\{|\varepsilon_{(\xi)} - \varepsilon_{(\xi-1)}|/\varepsilon_{(\xi)} \leq 10^{-6} \text{ or } \xi = 50\}$ **then** {Go to Step 4}
- 15: **else** {Set $\varepsilon_{(\xi)} = \varepsilon_{(\xi-1)}$ and Go to Step 2}

▷ **Step 4: Exit**

- 16: Compute the common priority vector $\tilde{\mathbf{q}}_{(g)} = \mathbf{q}_{(g)} / (\mathbf{1}^T \mathbf{q}_{(g)})$.

▷ **Step 5: End**

Algorithm 5 The AHP multi-components, multi-criteria, multi-groups and multi-actors decision problem: the MC-CPVP approach. Multi-Components problem.

▷ **Procedure for the CPVP problem of LEVEL 1:**

- G components ($g = 1, \dots, G$)
 - G weights $\pi_{(g)}$ for each component with $\sum_{g=1}^G \pi_{(g)} = 1$.
-

Input: The set of G CPVP priority vectors $\tilde{\mathbf{q}}_{(g)}$ of the g -th component

Output: the overall common preference vector $\tilde{\mathbf{q}}$.

▷ **Step 1: Starting step**

- 1: A set of G positive weights $\pi_{(g)}$ is chosen (with $\sum_{g=1}^G \pi_{(g)} = 1$);
- 2: The set of the matrices $\mathbf{W}_{(g)} = \tilde{\mu}_{(g)} \tilde{\mathbf{q}}_{(g)} \tilde{\mathbf{q}}_{(g)}^T$ is formed (with $g = 1, \dots, G$), where $\tilde{\mu}_{(g)}$ and $\tilde{\mathbf{q}}_{(g)}$ have been computed in Level 2;
- 3: All the salience $\tilde{\mu}_{(g)}$ are initialized to 1;
- 4: Set the iterative counter ξ to 1;
- 5: The common vector $\mathbf{q}$ is set to the first standardized eigenvector of $\sum_{g=1}^G \pi_{(g)} \mathbf{W}_{(g)}$ and let $\mu_{(g)}$ be the associated eigenvalue;
- 6: Compute each salience $\tilde{\mu}_{(g)} = \mathbf{q}^T \mathbf{W}_{(g)} \mathbf{q}$ with $g = 1, \dots, G$;
- 7: Compute $\eta = L_1(\mathbf{q}, \tilde{\mu}_{(g)}, \pi_{(g)})$; ▷ See loss function (7)
- 8: Set the convergence criterion variable $\varepsilon_{(\xi)}$ to η .

▷ **Step 2: Iterative step**

- 9: Increase the iterative counter $\xi = \xi + 1$;
- 10: The common vector $\mathbf{q}$ is set to the first standardized eigenvector of $\sum_{g=1}^G \pi_{(g)} \mathbf{W}_{(g)}$ and let $\mu_{(g)}$ be the associated eigenvalue;
- 11: Compute each salience $\tilde{\mu}_{(g)} = \mathbf{q}^T \mathbf{W}_{(g)} \mathbf{q}$ with $g = 1, \dots, G$;
- 12: Compute $\eta = L_1(\mathbf{q}, \tilde{\mu}_{(g)}, \pi_{(g)})$; ▷ See loss function (7)
- 13: Set the value of the convergence criterion variable $\varepsilon_{(\xi)}$ to η .

▷ **Step 3: Convergence step**

- 14: **if** $\{|\varepsilon_{(\xi)} - \varepsilon_{(\xi-1)}|/\varepsilon_{(\xi)} \leq 10^{-6} \text{ or } \xi = 50\}$ **then** {Go to Step 4}
- 15: **else** {Set $\varepsilon_{(\xi)} = \varepsilon_{(\xi-1)}$ and Go to Step 2}

▷ **Step 4: Exit**

- 16: Compute the overall common preference vector $\tilde{\mathbf{q}} = \mathbf{q}/(\mathbf{1}^T \mathbf{q})$.

▷ **Step 5: End**